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NASA Contractor Report 191500

ICASE Report No. 93-42

AD-A270 533

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SEP 06 1993

NASA Contract Nos. NAS1-18605, NAS1-19480  
July 1993

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Operated by the Universities Space Research Association



National Aeronautics and  
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Hampton, Virginia 23681-0001

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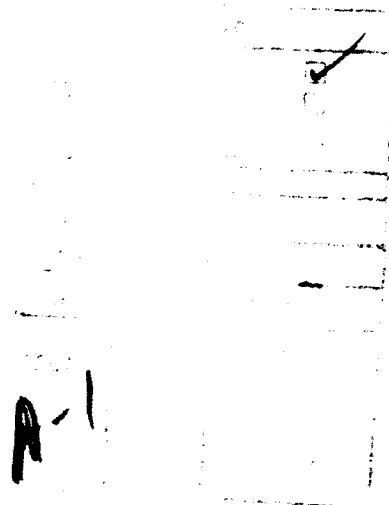
# PRECONDITIONING AND THE LIMIT TO THE INCOMPRESSIBLE FLOW EQUATIONS

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## Abstract

We consider the use of preconditioning methods to accelerate the convergence to a steady state for both the incompressible and compressible fluid dynamic equations. We also consider the relation between them for both the continuous problem and the finite difference approximation. The analysis relies on the inviscid equations. The preconditioning consists of a matrix multiplying the time derivatives. Hence, the steady state of the preconditioned system is the same as the steady state of the original system. For finite difference methods the preconditioning can change and improve the steady state solutions. An application to flow around an airfoil is presented.

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\*This research was partially supported by the National Aeronautics and Space Administration under NASA contract Nos. NAS1-18605 and NAS1-19480 while the first and third authors were in residence at the Institute for Computer Applications in Science and Engineering ( ICASE), NASA Langley Research Center, Hampton, Va. 23681

# 1 Introduction

Seymour Parter has considered preconditioning methods for numerical approximations to elliptic partial differential equations. As an extension of his ideas we shall consider similar techniques for the fluid dynamic equations. Much effort has been expended to solve the compressible steady state fluid equations for a large range of Mach numbers. A standard way of solving the steady state equations is to march the time dependent equations until a steady state is reached. Since the transient is not of any interest one can use acceleration techniques which might destroy the time accuracy but enables one to reach the steady state faster. Such methods can be considered as preconditionings to accelerate the convergence to a steady state. For the incompressible equations the continuity equation does not contain any time derivatives. To overcome this difficulty, Chorin [2] added an artificial time derivative of the pressure to the continuity equation together with a multiplicative variable,  $\beta$ . With this artificial term the resultant scheme is a symmetric hyperbolic system for the inviscid terms. Thus, the system is well posed and a numerical method for hyperbolic systems can be used to advance this system in time. The free parameter  $\beta$  is then chosen to reach the steady state quickly. Later Turkel ([15], [16]) extended this concept by adding a pressure time derivative also to the momentum equations. The resulting system after preconditioning is no longer symmetric but can be symmetrized by a change of variables.

It is well known, that it is difficult to solve the compressible equations for low Mach numbers. For an explicit scheme this is easily seen by inspecting the time steps. For stability, the time step must be chosen inversely proportional to the largest eigenvalue of the system which, for slow flows, is approximately the speed of sound,  $c$ . However, other waves are convected at the fluid speed,  $u$ , which is much slower. Hence, these waves don't change very much over a time step. Thus, thousands of time steps may be required to reach a steady state. Should one try a multigrid acceleration one finds that the same disparity in wave speeds slows down the multigrid acceleration. With an implicit method an ADI factorization is usually used so that one can easily invert the implicit factors. The use of ADI introduces factorization errors which again slows down the convergence rate when there are wave speeds of very different magnitudes [12].

For small Mach numbers it can be shown ([5],[7],[9]) that the incompressible equations approximate the compressible equations. Hence, one needs to justify the computational use of the compressible equations for low Mach flows. We present several reasons why one would still use the compressible equations even though the Mach number of the flow is small.

- There are many highly efficient compressible codes available that could be used for such problems especially in complicated geometries.
- For low speed aerodynamic problems at high angle of attack most of the flow consists of a low Mach number flow. However, there are localized regions containing shocks.
- In many problems thermal effects are important and the energy equation is coupled to the other equations. Then, the compressible equations must be used even for low Mach number flows.

Therefore, one wants to change the transient nature of the system to remove this disparity of the wave speeds. Based on an analogy with conjugate gradient methods such methods were called [15] preconditioned methods since the object is to reduce the condition number of the matrix. Another approach, in one dimension, is to diagonalize the matrix of the inviscid term. One can then use a different time step for each equation, or wave. Upon returning to the original variables one finds that this is equivalent to multiplying the time derivatives by a matrix. Hence, this same approach

is named characteristic time stepping in [17]. In multidimensions one can no longer completely decouple the waves and so the characteristic time stepping is only an approximation.

Thus, for both the incompressible and compressible equations we will consider systems of the form

$$w_t + f_x + g_y = 0.$$

This system is written in conservation form though for some applications this is not necessary. Our analysis will be based on the linearized equations so that the conservation form does not appear in the analysis though it does appear in the final numerical approximation. This system is now replaced by

$$\mathbf{P}^{-1}w_t + f_x + g_y = 0,$$

or in linearized form

$$\mathbf{P}^{-1}w_t + Aw_x + Bw_y = 0, \quad (1)$$

with A and B constant matrices.

In order for this system to be equivalent to the original system, in the steady state, we demand that  $\mathbf{P}^{-1}$  have an inverse. This only need be true in the flow regime under consideration. We shall see later that frequently  $\mathbf{P}$  is singular at stagnation points and also along sonic lines. Thus, we will temporarily consider strictly subsonic flow without a stagnation point. For transonic flow it is necessary to smooth out the singularity in a neighborhood of the sonic line. We also assume that the Jacobian matrices  $A = \frac{\partial f}{\partial w}$  and  $B = \frac{\partial g}{\partial w}$  are simultaneously symmetrizable. In terms of the 'symmetrizing' variables we also demand that  $\mathbf{P}$  be positive definite. We shall show later, in detail, that it does not matter which set of dependent variables are used to develop the preconditioner. One can transform between any two sets of variables. Popular choices are two out of density, pressure, enthalpy, entropy or temperature in addition to the velocity components. Thus, when we are finished we will analyze a system which is similar to (1), where the matrices A and B are symmetric and  $\mathbf{P}$  is both symmetric and positive definite. Such systems are known as symmetric hyperbolic systems. One can then multiply this system by  $w$  and integrate by parts to get estimates for the integral of  $w_t^2$ , i.e. energy estimates. These estimates can then be used to show that the system is well posed (see e.g. [5]). We stress that if  $\mathbf{P}$  is not positive then we may change the physics of the problem. For example, if  $\mathbf{P} = -I$  then we have reversed the time direction and must therefore change all the boundary conditions. Hence, to be sure that the system is well posed with the original type of boundary conditions we shall only consider the symmetric hyperbolic system.

With these assumptions the steady state solutions of the two systems are the same. Assuming the steady state has a unique solution, it does not matter which system we march to a steady state. We shall later see that for the finite difference approximations the steady state solutions are not necessarily the same and usually the preconditioned system leads to a better behaved steady state.

## 2 Incompressible equations

We first consider the incompressible inviscid equations in primitive variables.

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vv_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0 \end{aligned}$$

We consider generalizations of Chorin's pseudo-compressibility method [2]. Using the preconditioning suggested in [15] (with  $\alpha = 1$ ) we have

$$\begin{aligned}\frac{1}{\beta^2}p_t + u_x + v_y &= 0 \\ \frac{u}{\beta^2}p_t + u_t + uu_x + vu_y + p_x &= 0 \\ \frac{v}{\beta^2}p_t + v_t + uv_x + vv_y + p_y &= 0\end{aligned}\quad (2)$$

or in conservation form

$$\begin{aligned}\frac{1}{\beta^2}p_t + u_x + v_y &= 0 \\ \frac{2u}{\beta^2}p_t + u_t + (u^2 + p)_x + (uv)_y &= 0 \\ \frac{2v}{\beta^2}p_t + v_t + (uv)_x + (v^2 + p)_y &= 0\end{aligned}$$

We can also write (2) in matrix form using

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \beta^2 & 0 & 0 \\ -u & 1 & 0 \\ -v & 0 & 1 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ 1 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 1 \\ 0 & v & 0 \\ 1 & 0 & v \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \quad (3)$$

Multiplying by  $\mathbf{P}$  we rewrite this as

$$w_t + \mathbf{P}Aw_x + \mathbf{P}Bw_y = 0.$$

We also define

$$D = \omega_1 A + \omega_2 B \quad -1 \leq \omega_1, \omega_2 \leq 1$$

where  $\omega_1, \omega_2$  are the Fourier transform variables in the  $x$  and  $y$  directions respectively. The speeds of the waves are now governed by the roots of  $\det(\lambda I - \mathbf{P}A\omega_1 - \mathbf{P}B\omega_2) = 0$  or equivalently  $\det(\lambda \mathbf{P}^{-1} - A\omega_1 - B\omega_2) = 0$ . Let

$$q = u\omega_1 + v\omega_2.$$

Then the eigenvalues of  $\mathbf{P}D$  are

$$\begin{aligned}d_0 &= q \\ d_{\pm} &= \pm\beta\end{aligned}\quad (4)$$

and so the 'acoustic' speed is isotropic.

The spatial derivatives involve symmetric matrices, i.e.  $D$  is a symmetric matrix but  $P$  is not symmetric. Thus, while the original system was symmetric hyperbolic the preconditioned system is no longer symmetric. In [15] it is shown that as long as

$$\beta^2 > (u^2 + v^2)$$

then the equations can be symmetrized. On the other hand the eigenvalues are most equalized if  $\beta^2 = (u^2 + v^2)$  [15]. Hence, we wish to choose  $\beta^2$  slightly larger than  $u^2 + v^2$ . However, numerous calculations verify, that in general, a constant  $\beta$  is the best for the convergence rate. The reasons for this are not clear.

We wish to stress that  $\beta$  has the dimensions of a speed. Therefore,  $\beta$  cannot be a universal constant. There are papers that claim that  $\beta = 1$  or  $\beta = 2.5$  are optimal. Such claims cannot be true in general. It is simple to see that if one nondimensionalizes the equation then  $\beta$  gets divided by a reference velocity. Hence, the optimal 'constant'  $\beta$  depends on the dimensionalization of the problem and in particular depends on the inflow conditions. In many calculations the inflow mass flux is equal to 1 or else  $p + (u^2 + v^2)/2 = 1$ . Such conditions will give an optimal  $\beta$  close to one. However, if one chooses the incoming mass flux as ten then the optimal  $\beta$  will be larger.

We next define the Bernoulli function

$$H = p + (u^2 + v^2)/2.$$

Bernoulli's theorem states that for steady inviscid flow  $H$  is constant along streamlines. We now multiply the second equation of (2) by  $u$  and the third equation of (2) by  $v$  and add these two equations. If  $\beta^2 = u^2 + v^2$ , the result is

$$H_t + uH_x + vH_y = 0. \quad (5)$$

Thus, by altering the time dependence of the equations we have constructed a new equation in which  $H$  is convected along streamlines. Furthermore, if  $H$  is a uniform constant both initially and at inflow then  $H$  will remain constant for all time. On the numerical level this will usually not be true because of the introduction of an artificial viscosity or because of upwinding. For viscous flow, (5) is replaced by

$$H_t + uH_x + vH_y = \frac{1}{Re}(u\Delta u + v\Delta v)$$

We note that these relationships for  $H$  follow from the momentum equations and do not depend on the form of the continuity equation. Hence, we consider the following generalization of (2)

$$\begin{aligned} \frac{1}{\beta^2} p_t + aH_t + u_x + v_y &= 0 \\ \frac{\alpha u}{\beta^2} p_t + u_t + uu_x + vv_y + p_x &= 0 \\ \frac{\alpha v}{\beta^2} p_t + v_t + uv_x + vv_y + p_y &= 0 \end{aligned}$$

where,  $a$  is a free parameter. The eigenvalues of  $PD$  are independent of the parameter  $a$  and are given by (4). For  $a = 0, \alpha = 1$  we recover our original scheme. For  $a = -1$  the time derivative of the pressure no longer appears in the continuity equation. For general  $\beta$  we have

$$\mathbf{P}^{-1} = \frac{1}{\beta^2} \begin{pmatrix} (a+1) & au & av \\ \alpha u & \beta^2 & 0 \\ \alpha v & 0 & \beta^2 \end{pmatrix},$$

$$\mathbf{P} = \frac{1}{d} \begin{pmatrix} \beta^2 & -au & -av \\ -\alpha u & 1 + a - \frac{a\alpha v^2}{\beta^2} & \frac{a\alpha uv}{\beta^2} \\ -\alpha v & \frac{a\alpha uv}{\beta^2} & 1 + a - \frac{a\alpha u^2}{\beta^2} \end{pmatrix}$$

where  $d = 1 + a - a\alpha \frac{u^2+v^2}{\beta^2}$  and we require that  $d \geq 0$ . If  $\beta^2 = u^2 + v^2$  and  $\alpha = 1$  then

$$\mathbf{P} = \begin{pmatrix} u^2 + v^2 & -au & -av \\ -u & 1 + \frac{au^2}{u^2+v^2} & \frac{auv}{u^2+v^2} \\ -v & \frac{auv}{u^2+v^2} & 1 + \frac{av^2}{u^2+v^2} \end{pmatrix}$$

In [16] an analogy to the symmetric preconditioning of van Leer, Lee and Roe was constructed for the incompressible equations. If we choose  $a = 1, \alpha = 1$  we get this preconditioning of van Leer et.al., i.e.  $\mathbf{P}$  is symmetric.

These examples show that the preconditioning is not unique. In fact, since the determinant of the transpose of a matrix is equal to the determinant of the original matrix it follows that the transpose of  $\mathbf{P}$  is also a preconditioner with the same eigenvalues for the preconditioned system. In general, these various systems will have similar eigenvalues but different eigenvectors for the preconditioned system. Numerous calculations show that the system given by  $\mathbf{P}$  in (2) is more robust and converges faster than that with the transpose preconditioner. This shows that it is not sufficient to consider just the eigenvalues but that the eigenvectors are also of importance. However, even when  $\mathbf{P}$  is symmetric  $\mathbf{PD}$  is not symmetric and so the eigenvectors of the preconditioned system do not form an orthogonal basis.

We next examine some general form that the preconditioner can have. For this analysis it is easier to use streamwise coordinates as suggested in [17] and so  $v = 0$ . Let  $u_*$  be some normalization of the velocity components, then

$$A = \begin{pmatrix} 0 & u_* & 0 \\ u_* & u & 0 \\ 0 & 0 & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & u_* \\ 0 & 0 & 0 \\ u_* & 0 & 0 \end{pmatrix}$$

Then the "convective" eigenvector for the non-preconditioned system is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{-u\omega_1 + \sqrt{(u\omega_1)^2 + 4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{-u\omega_1 - \sqrt{(u\omega_1)^2 + 4u_*^2}}{2} \\ u_*\omega_1 \\ u_*\omega_2 \end{pmatrix}$$

We now consider preconditioners of the form

$$\mathbf{P} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

Let  $D = \omega_1 A + \omega_2 B$ ,  $\omega_1^2 + \omega_2^2 = 1$ . We want the eigenvalues of  $\mathbf{P}D$  to be  $\omega_1 u$ ,  $\pm u$ . This gives us three relations for the four unknowns:

$$a = \frac{u^2}{u_*^2}$$

$$(b + c)u_* + du = 0$$

$$u^2 d - bcu_*^2 = u^2$$

The values suggested in [15] are  $b = 0, c = -\frac{u}{u_*}, d = 1$  while the values suggested in [17] are  $b = c = -\frac{u}{u_*}, d = 2$ . We next present the eigenvectors of  $\mathbf{P}D$  in terms of the elements of  $\mathbf{P}$ . We exclude the case  $\omega_2 = 0, \omega_1 = \pm 1$  as in this case  $\mathbf{P}D$  has a double eigenvalue  $u$  and the eigensystem completely changes. Then the "convective" eigenvector is

$$\begin{pmatrix} 0 \\ \omega_2 \\ -(1 + b\frac{u_*}{u})\omega_1 \end{pmatrix}.$$

The "acoustic" eigenvectors are given by

$$\begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 - \frac{u^2\omega_1}{u_*} + bu\omega_1 \\ u_*(a + b\omega_1^2) - u_*(b + c)\omega_1 \\ (u_*b\omega_1 + u)\omega_2 \end{pmatrix}, \quad \begin{pmatrix} \frac{u^2}{u_*} - bu\omega_1^2 + \frac{u^2\omega_1}{u_*} - bu\omega_1 \\ u_*(a + b\omega_1^2) + u_*(b + c)\omega_1 \\ (u_*b\omega_1 - u)\omega_2 \end{pmatrix}.$$

We note that the convective eigenvector is the same as before the preconditioning for the choice  $b = 0$ . The two acoustic eigenvectors are orthogonal to each other if we choose  $b = 0$  and  $c^2 = \frac{u^2(1 - \frac{u^2}{u_*^2})}{u_*^2}$ . This is similar, but not identical, to the choice suggested in [15]. There is no way to make the convective eigenvector normal to both acoustic eigenvectors for preconditioners of the form (6).

### 3 Compressible equations

The time dependent Euler equations can be written as

$$\begin{aligned} p_t + up_x + vp_y + \rho a^2(u_x + v_y) &= 0 \\ u_t + uu_x + vu_y + \frac{p_x}{\rho} &= 0 \\ v_t + uv_x + vv_y + \frac{p_y}{\rho} &= 0 \\ S_t + uS_x + vS_y &= 0 \end{aligned} \quad (7)$$

where  $a$  is the speed of sound given by  $a^2 = \frac{\gamma p}{\rho}$ .

The form of this system is unchanged if we nondimensionalize the equations. From now on we shall assume that  $u, v, p, \rho$  are nondimensional quantities where the dimensional variables are



nondimensionalized by  $u_*, p_*, \rho_*$  with  $p_* = \rho_* u_*^2$ . Following [5] we define  $\epsilon = \frac{u_*}{a_*}$ . If the fluid is isentropic then

$$p = \frac{\rho^\gamma}{\gamma \epsilon^2} \quad (8)$$

and

$$a = \frac{\rho^{\frac{\gamma-1}{2}}}{\epsilon} \quad (9)$$

Hence, as  $\epsilon$  goes to zero the speed of sound,  $a$ , goes to infinity and so the first equation in (7) reduces to  $u_x + v_y = 0$ .

It was pointed out in ([15], [16]) that these equations can be symmetrized by using  $\frac{dp}{\rho a}$  as the independent variable rather than  $dp$ . Hence, we define a new variable  $\phi$  by  $d\phi = \frac{dp}{\rho a}$ . For isentropic flow both  $p$  and  $a$  are functions only of the density and so using (8, 9) this can be integrated explicitly. This gives  $\phi = \frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}\epsilon}$ . As the Mach number goes to zero  $\phi$  tends to infinity and therefore, Gustafsson and Stoor [5] subtract a constant and define

$$\phi = \frac{\rho^{\frac{\gamma-1}{2}} - 1}{\frac{\gamma-1}{2}\epsilon}.$$

This amounts to specifying the constant in the integration of  $d\phi$  from  $dp$ . They then prove, using energy methods, that for the linearized equations

$$a\phi_x \rightarrow \frac{\partial p_{\text{incompressible}}}{\partial x}$$

Since  $\rho \rightarrow 1$  and using the definition of  $d\phi$  this is equivalent to

$$dp_{\text{compressible}} \rightarrow dp_{\text{incompressible}}. \quad (10)$$

We consider preconditionings that are a generalization of (3)

$$\begin{pmatrix} \frac{a^2}{\beta^2} & 0 & 0 & 0 \\ \frac{\alpha a u}{\beta^2} & 1 & 0 & 0 \\ \frac{\alpha a v}{\beta^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_t + \begin{pmatrix} u & a & 0 & 0 \\ a & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_x + \begin{pmatrix} v & 0 & a & 0 \\ 0 & v & 0 & 0 \\ a & 0 & v & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \begin{pmatrix} \frac{dp}{\rho a} \\ du \\ dv \\ dS \end{pmatrix}_y = 0$$

The nonpreconditioned case corresponds to  $\beta^2 = a^2, \alpha = 0$ . Let

$$q = u\omega_1 + v\omega_2$$

then the eigenvalues of PD are given by

$$d_0 = q \quad (\text{double})$$

$$d_{\pm} = 1/2 \left[ (1 - \alpha + \beta^2/a^2)q \pm \sqrt{((1 - \alpha + \beta^2/a^2)^2 q^2 + 4(1 - q^2/a^2)\beta^2)} \right]$$

If we consider the special case  $\alpha = 1 + \beta^2/a^2$  we find that the 'acoustic' eigenvalues are given by

$$d_{\pm} = \sqrt{(1 - q^2/a^2)\beta^2}$$

Hence, these eigenvalues are isotropic in the limit of  $M$  going to zero. However, this eigenvalue vanishes at the sonic line and so the matrix is singular. In general if we demand that the acoustic eigenvalues be isotropic then we have a singularity at the sonic line where the eigenvalues cannot be isotropic. The two ways out of this difficulty are either to smooth the formulas near the singular line or else to give up on isotropy. This difficulty is not a property of the preconditioning just presented but applies equally to all preconditioners.

We now consider the system (7) in conservation form.

$$\begin{aligned}\rho_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0 \\ E_t + (\rho H u)_x + (\rho H v)_y &= 0\end{aligned}$$

where

$$\begin{aligned}E &= \frac{p}{\gamma - 1} + \frac{u^2 + v^2}{2} \\ H &= \frac{E + p}{\rho} = \frac{a^2}{\gamma - 1} + \frac{u^2 + v^2}{2}.\end{aligned}$$

Note that the Bernoulli function  $H$  is not identical with  $H$  for the incompressible equations. However, we again have that for steady inviscid flow  $H$  is constant along stream lines. We now precondition the density and the energy equations in the following consistent manner. Let  $\psi$  be any variable we choose. Then we consider

$$\begin{aligned}\psi_t + (\rho u)_x + (\rho v)_y &= 0 \\ (\psi H)_t + (\rho H u)_x + (\rho H v)_y &= 0\end{aligned}$$

Manipulating these equations gives

$$H_t + u H_x + v H_y = 0$$

i.e. the total enthalpy,  $H$ , is simply convected in time along streamlines as we obtained for  $H$  in the incompressible case. It is interesting to observe that in the incompressible case we achieved this by preconditioning only the momentum equations while for the compressible flow we achieve this by preconditioning the continuity and energy equation. Of course, for isentropic flow the energy equation is not independent of the other equations and the result is not surprising.

For the finite difference equation this implies that the artificial viscosity for the continuity equation should be based on  $\psi$  and for the energy equation on  $\psi H$ . If we choose  $\psi = \rho$ , i.e. no preconditioning for the continuity equation then we have the same artificial viscosity as suggested in [6] but with a different variable being advanced in time. If we choose  $\psi = p$  then both the continuity and energy equations are preconditioned.

We next present the van Leer-Lee-Roe preconditioning for general non-aligned flow in  $(\frac{dp}{\rho a}, du, dv, dS)$  variables [17].

$$\mathbf{P}_V = \begin{pmatrix} \frac{\tau}{\beta^2} M^2 & -\frac{\tau}{\beta^2} u/a & -\frac{\tau}{\beta^2} v/a & 0 \\ -\frac{\tau}{\beta^2} u/a & (\frac{\tau}{\beta^2} + 1) \frac{u^2}{u^2 + v^2} + \tau \frac{v^2}{u^2 + v^2} & (\frac{\tau}{\beta^2} + 1) \frac{uv}{u^2 + v^2} & 0 \\ -\frac{\tau}{\beta^2} v/a & (\frac{\tau}{\beta^2} + 1) \frac{uv}{u^2 + v^2} & (\frac{\tau}{\beta^2} + 1) \frac{v^2}{u^2 + v^2} + \tau \frac{u^2}{u^2 + v^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\beta = \begin{cases} \sqrt{1-M^2}, & M < 1, \\ \sqrt{M^2-1}, & M \geq 1; \end{cases}$$

$$\tau = \begin{cases} \sqrt{1-M^2}, & M < 1, \\ \sqrt{1-M^{-2}}, & M \geq 1. \end{cases}$$

At the sonic line  $\beta = 0$  and  $\tau = 0$  and the preconditioning matrix becomes singular. This preconditioning is not unique even if one only considers symmetric preconditioners. In both these examples the preconditioner was constructed based on using  $(p, u, v, S)$  as the dependent variables. The reason for this choice is that the matrices are symmetric with this choice. However, if another choice of variables is more appropriate that introduces no difficulties. Thus, for example [1] recommends the use of  $(p, u, v, T)$  variables for the Navier-Stokes equations. Given two sets of dependent variables  $w$  and  $W$  let  $W_w$  be the Jacobian matrix  $\frac{\partial W}{\partial w}$ . Then, we have  $dW = W_w dw$ . So we can go between any sets of primitive variables or between primitive variables and conservation variables. In particular since the equations are solved in conservation variables we have several ways of going from the primitive variable preconditioner to a conservation variable preconditioner. Thus, the choice of variables used in constructing the preconditioner is dictated by mathematical or physical reasoning and then the preconditioner can be transformed to any other set of variables.

- Construct the preconditioner matrix for the conservation variables. If  $W$  are the conservative variables and  $w$  the primitive variables then  $P_{conservative} = (W_w)^{-1} P_{primitive}(W_w)$ . Details of the matrix Jacobians between various sets of variables are given in the appendix.
- We calculate the residual  $dW$  in conservative variables. We then transform  $dW$  to  $dw$  as before. Next we multiply by  $P$  and finally transform back to conservative variables  $dW$  and update the solution. This is algebraically equivalent to the first option but requires three matrix multiplies instead of one. However, it offers more flexibility.
- Similar to the previous suggestion we calculate the residual  $dW$  and transform to conservative variables  $dw$  and then multiply by  $P$ . At this stage we update the primitive variables  $w$ . We then use the nonlinear relations to construct  $W$  from  $w$ . This approach has advantages if the boundary conditions are given in terms of the primitive variables ( $p$  or  $T$ ) and so they can be specified exactly and not approximately.

If the residual  $dW$  is kept from the conservation form but the time derivative  $W_t$  is replaced by the time derivative of other variables,  $\tilde{W}_t$  this is linearly equivalent to preconditioning by the matrix  $P^{-1} = \frac{\partial W}{\partial \tilde{W}}$ .

These methods are all equivalent for linear systems and the difference between them is mainly one of convenience. However, we shall next see that for the difference approximation these approaches are not equivalent.

## 4 Difference Equations

Until now the entire analysis has been based on the partial differential equation. We now make some remarks on important points for any numerical approximation of this system.

- For an upwind difference scheme based on a Riemann solver this Riemann solver should be for the preconditioned system and not the original scheme. In [3] plots are shown to illustrate

the greatly improved accuracy for low Mach number flows when the Riemann solver is based on the preconditioning.

- For central difference schemes there is a need to add an artificial viscosity. Accuracy is improved for low Mach number flows if the preconditioner is applied only to the physical convective and viscous terms but not to the artificial viscosity. Volpe [19] shows that the accuracy of the original system deteriorates as the Mach number is reduced. The use of a matrix artificial dissipation ([14]) should be based on the preconditioned equations as in the upwind difference scheme. Upwind schemes without preconditioning tend to have difficulties with accuracy for low Mach flows [3].

Hence, both for upwind and central difference schemes the Riemann solver or artificial viscosity should be based on  $\mathbf{P}^{-1}|\mathbf{P}\mathbf{A}|$  and not  $|\mathbf{A}|$ , i.e. in one dimension solve  $w_t + \mathbf{P}f_x = (|\mathbf{P}\mathbf{A}|w_x)_x$ . For a scalar artificial viscosity  $|\mathbf{P}\mathbf{A}|$  is replaced by the spectral radius of  $\mathbf{P}\mathbf{A}$  or equivalently the time step associated with the preconditioned matrix. This is equivalent to not multiplying the artificial viscosity by  $\mathbf{P}$ .

- For a central difference scheme with a scalar artificial viscosity the artificial viscosity is of the form of a high order difference of the same quantity as is advanced in time. Thus, the continuity equation is solved for the density and so the artificial viscosity is a difference of the density. Similarly, for the momentum equations. For, the energy equation one can base the artificial viscosity on the energy. Alternatively it can be based on the total enthalpy which guarantees, for inviscid flow, that the total enthalpy is constant in the steady state [6]. When preconditioning the system one of the alternatives described above was to replace the time derivative of the conservative quantities with the time derivative of other variables. This, implies that the artificial viscosity should also be changed. Thus, if the continuity equation is updated for the pressure rather than the density, then the artificial viscosity should be based on the pressure. This is physically more reasonable for low speed flow since the density is almost constant and so will not contribute any reasonable viscosity. Furthermore, using a viscosity in the continuity equation based on the pressure mimics what was done for the incompressible equations. This allows the low speed compressible equations to replicate the results of the incompressible equations on the finite difference level. This will be discussed in more detail in the following sections.

- When using characteristics in the boundary conditions these should be based on the characteristics of the modified system and not the physical system.
- When using multigrid it is better to transfer the residuals based on the preconditioned system to the next grid since these residuals are more balanced than the physical residuals.

Preconditioning is even more important when using multigrid than with an explicit scheme. With the original system the disparity of the eigenvalues greatly affects the smoothing rates of the slow components and so slows down the multigrid method, [10].

- As indicated above there are accuracy difficulties at low Mach numbers [19]. Some of these can be alleviated by preconditioning the dissipation terms. For very small Mach numbers there is also a difficulty with roundoff errors as  $\frac{p}{u^2+v^2} \rightarrow \infty$ . Several people have suggested subtracting out a constant pressure from the dynamic pressure. A more detailed analysis [4] suggests replacing the pressure  $p$  by  $\bar{p}$  where  $p = \frac{\rho_0}{2} \bar{p}$  and  $\epsilon$  is a representative Mach number.

We conclude from the above remarks that the steady state solution of the preconditioned system may be different from that of the physical system. Thus, on the finite difference level the preconditioning can improve the accuracy as well as the convergence rate.

In the previous section we stated that it is not important if one updates a different set of variables or else uses the conservation variables and compensates with preconditioning by a matrix multiplication. However, numerically for very small Mach numbers the entries in the preconditioning matrix can become very large or small. Hence, it can be advantageous to update the pressure or temperature directly rather than using a matrix multiply for preconditioning.

## 5 Convergence

We have previously quoted several papers ([5],[7], [9]), that prove the convergence of the compressible equations to the incompressible equations, for isentropic flow, as the Mach number goes to zero. For nonisentropic flow there are no formal proofs. However, it is clear that for viscous flows that the boundary condition on the temperature, adiabatic or isothermal is very important, see [11].

All these results refer to the time dependent physical equations. Once preconditioning is introduced time accuracy is lost and one can only discuss convergence of the steady state solutions. In this case one would hope that the time dependent preconditioned compressible equations converge to some time dependent preconditioning of the incompressible equations. In addition, one would also want this to be true on the numerical level. Thus, one would want to solve the preconditioned compressible equations by some numerical technique, on a fixed mesh and compare that with the solution of the incompressible equations on the same mesh. Mathematically, we have two limit processes occurring: the Mach number going to zero and the mesh size going to zero. These two limits need not commute. If one first converges the mesh size and then the Mach number it is equivalent to the convergence proofs for the analytic case. The more interesting problem is to converge the Mach number and then converge the mesh, i.e. we use a fix mesh as the Mach number is reduced.

In particular this requires a careful study of the viscosities introduced by the scheme. We first consider an upwinding scheme. For the compressible case we have already noted that the Riemann solver should depend on the preconditioned problem. One would then need to show that this Riemann problem converges to a Riemann problem for some preconditioning of the incompressible equations. We next consider a central difference scheme with a scalar viscosity. In this case a high order even difference of some quantity is added separately to each equation, e.g. for the incompressible equations: pressure for the continuity equation,  $u$  and  $v$  for the momentum equations. For the compressible equations one normally adds a density difference to the continuity equation. In such a case it is obvious that the numerical scheme for the compressible equations cannot converge to the numerical scheme for the incompressible equations. Furthermore, for low Mach number flows the density is almost constant and so the higher order difference of the density does not add much of a viscosity to the continuity equation. As such, we conclude that the artificial viscosity for the compressible continuity equation should be based on pressure and not density (at least for low Mach numbers).

We shall examine the convergence a little more closely. By convergence of the compressible equations to the incompressible equations we are merely verifying what happens to the difference equations as the Mach number goes to zero. The convergence of the solution of the numerical approximation to the preconditioned compressible equations to the numerical solution of the incompressible equation is more difficult. However, we shall see that for the numerical solution the convergence of the difference equation is nontrivial and depends on the preconditioning. For this

purpose we shall only consider a central difference approximation together with a scalar artificial viscosity for the nondimensionalized preconditioned inviscid equations.

For the incompressible equations in nonconservative form we consider the preconditioned system

$$\begin{aligned} p_t + \beta^2(u_x + v_y) &= h^3[(K_1 p_{xxx})_x + (K_2 p_{yyy})_y] \\ \frac{u}{\beta^2} p_t + u_t + uu_x + vu_y + p_x &= h^3[(K_1 u_{xxx})_x + (K_2 u_{yyy})_y] \\ \frac{v}{\beta^2} p_t + v_t + uv_x + vv_y + p_y &= h^3[(K_1 v_{xxx})_x + (K_2 v_{yyy})_y] \end{aligned} \quad (11)$$

where each space derivative is approximated by a central difference with spacing  $h$  in each direction. The time derivatives are replaced by a multi-stage scheme.  $K_1, K_2$  are the largest eigenvalues of the coefficient matrix in the respective direction. Since we do not expect shocks we only consider a linear fourth difference artificial viscosity and not a nonlinear second difference [6], see the result section for more details.

We next consider the same scheme for the preconditioned compressible inviscid equations, under the assumption that the entropy,  $S$ , is constant so that  $p = p(\rho)$ . It is easier to analyze the convergence for the nonsymmetric form since the pressure,  $p$ , converges and not  $\frac{dp}{d\rho}$ , see (10). For the preconditioned continuity equation we have

$$p_t + \frac{\beta^2}{a^2} [up_x + vp_y + \rho a^2(u_x + v_y)] = h^3[(K_1 p_{xxx})_x + (K_2 p_{yyy})_y]$$

Since  $p_x = a^2 \rho_x$ ,  $p_y = a^2 \rho_y$  we can rewrite the system as

$$\begin{aligned} p_t + \beta^2[(\rho u)_x + (\rho v)_y] &= h^3[(K_1 p_{xxx})_x + (K_2 p_{yyy})_y] \\ \frac{u}{\beta^2} p_t + u_t + uu_x + vu_y + \frac{p_x}{\rho} &= h^3[(K_1 u_{xxx})_x + (K_2 u_{yyy})_y] \\ \frac{v}{\beta^2} p_t + v_t + uv_x + vv_y + \frac{p_y}{\rho} &= h^3[(K_1 v_{xxx})_x + (K_2 v_{yyy})_y] \end{aligned} \quad (12)$$

Comparing (11) with (12) it is obvious that if  $\rho \rightarrow 1$  as  $M \rightarrow 0$  then (11) converges to (12). It is crucial for both the time derivative and the artificial viscosity in the compressible continuity equation to be pressure based rather than density based. The preconditioning of the momentum equations is not important for this convergence.

For the incompressible equations in conservative form we multiply the first equation in (11) by  $u$  and add it to the second and third equations. However, we do not change the artificial viscosity. Then

$$\begin{aligned} p_t + \beta^2(u_x + v_y) &= h^3[(K_1 p_{xxx})_x + (K_2 p_{yyy})_y] \\ \frac{2u}{\beta^2} p_t + u_t + (u^2 + p)_x + (uv)_y &= h^3[(K_1 u_{xxx})_x + (K_2 u_{yyy})_y] \\ \frac{2v}{\beta^2} p_t + v_t + (uv)_x + (v^2 + p)_y &= h^3[(K_1 v_{xxx})_x + (K_2 v_{yyy})_y] \end{aligned} \quad (13)$$

For the compressible equations in conservative form we have two choices. One choice is to multiply the first equation in (11) by  $u$  and the second by  $\rho$  and add the two. The spatial derivatives are then in conservation form. However, the time derivative is of the form  $\rho u_t$  rather than  $(\rho u)_t$ .

and the artificial viscosity terms are not in conservation form. Hence, we instead choose to apply the preconditioning directly to the conservative form. The resultant preconditioned compressible equations in conservative form is

$$\begin{aligned} p_t + \beta^2 [(\rho u)_x + (\rho v)_y] &= h^3 [(K_1 p_{xxx})_x + (K_2 p_{yyy})_y] \\ \frac{2u}{\beta^2} p_t + (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= h^3 [(K_1 (\rho u)_{xxx})_x + (K_2 (\rho u)_{yyy})_y] \\ \frac{v}{\beta^2} p_t + (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= h^3 [(K_1 (\rho v)_{xxx})_x + (K_2 (\rho v)_{yyy})_y] \end{aligned} \quad (14)$$

Note that (14) is not equivalent to (12).

In this case we again see that (14) converges formally to (13) as  $M \rightarrow 0$  and  $\rho \rightarrow 1$ . This is because the pressure is used for the time derivative and the artificial viscosity in the continuity equation.

This all applies to the isentropic equations. The compressible equations for nonisentropic flow is more complicated and in fact there does not exist any proof of the convergence of the solution of the compressible equations to the solution of the incompressible equations for this case.

## 6 Computational Results

We now present a calculation for two dimensional flow around an airfoil to demonstrate the previous theory. As described above the discretization is based on the multistage time method coupled with a central difference approximation as described in [6].

We solve the equation in conservation form based on a hybrid set of variables of those previously considered.

$$\begin{aligned} W_t + P(F_x + G_y) &= AD = (K_1 Q_{xxx})_x + (K_2 Q_{yyy})_y \\ W &= \begin{pmatrix} p' \\ \rho u \\ \rho v \\ E' \end{pmatrix} \\ F &= \begin{pmatrix} \rho u \\ \rho u^2 + p' \\ \rho uv \\ \rho H' u \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p' \\ \rho H' v \end{pmatrix}, \quad Q = \begin{pmatrix} p' \\ \rho u \\ \rho v \\ H' \end{pmatrix} \end{aligned}$$

where  $p' = p - p_\infty$ ,  $E' = c_p \rho (T - T_\infty) - (p - p_\infty) + \frac{\rho(u^2 + v^2)}{2}$  and  $\rho H' = E' + p'$ . We subtract the constants to keep the quantities in scale, see (10).

$$P = \Delta \cdot \begin{pmatrix} 1 & -\frac{u}{G+h_\infty} & -\frac{v}{G+h_\infty} & \frac{1}{G+h_\infty} \\ -B_2 & 1 + \frac{uB_2}{G+h_\infty} & \frac{vB_2}{G+h_\infty} & -\frac{B_2}{G+h_\infty} \\ -B_3 & \frac{uB_3}{G+h_\infty} & 1 + \frac{vB_3}{G+h_\infty} & -\frac{B_3}{G+h_\infty} \\ -B_4 & \frac{uB_4}{G+h_\infty} & \frac{vB_4}{G+h_\infty} & 1 - \frac{B_4}{G+h_\infty} \end{pmatrix}$$

where  $h = c_p T$ ,  $G = \frac{u^2 + v^2}{2}$ ,  $\Delta = \frac{(G+h_\infty)\beta^2}{h}$ ,

$$B_1 = \frac{1}{\beta^2} - \frac{1}{(\gamma - 1)h}$$

$$\begin{aligned}
B_2 &= B_1 u + \frac{\alpha u}{\beta^2} \\
B_3 &= B_1 v + \frac{\alpha v}{\beta^2} \\
B_4 &= B_1 H + \frac{\alpha(u^2 + v^2)}{\beta^2}
\end{aligned}$$

We choose

$$\beta^2 = \max(u^2 + v^2, 0.9(u_\infty^2 + v_\infty^2)), \quad \alpha = 1$$

These equations are given for the nondimensionalized variables. The nondimensionalization affects the convergence. In some codes,  $p$  and  $\rho$  are fixed in the far field. This implies that the speed of sound,  $a$ , is also bounded. As the Mach number goes to zero the pressure remains of order 1 while the velocities go to zero. Alternatively, one can nondimensionalize so that the velocities are of order 1 in the far field and then the pressure and speed of sound go to infinity, unless one subtracts an appropriate constant,

A typical step of a Runge-Kutta approximation is

$$W^{(k)} = W^{(0)} - \alpha_k \Delta t [D_x F^{(k-1)} + D_y G^{(k-1)} - AD],$$

where  $D_x$  and  $D_y$  are spatial differencing operators, and  $AD$  represents the artificial dissipation terms. The dissipation terms are a blending of second and fourth differences. That is,

$$AD = (D_x^2 + D_y^2 - D_x^4 - D_y^4) Q, \quad (15)$$

where

$$\begin{aligned}
D_x^2 Q &= \nabla_x \left[ \left( \lambda_{i+\frac{1}{2},j} \epsilon_{i+\frac{1}{2},j}^{(2)} \right) \Delta_x \right] Q_{i,j}, \\
D_x^4 Q &= \nabla_x \left[ \left( \lambda_{i+\frac{1}{2},j} \epsilon_{i+\frac{1}{2},j}^{(4)} \right) \Delta_x \nabla_x \Delta_x \right] Q_{i,j},
\end{aligned}$$

and  $\Delta_x, \nabla_x$  are the standard forward and backward difference operators respectively associated with the  $x$  direction. The variable scaling factor  $\lambda$  is chosen as

$$\lambda_{i+\frac{1}{2},j} = \frac{1}{2} [(\bar{\lambda}_x)_{i,j} + (\bar{\lambda}_x)_{i+1,j}]$$

where  $\bar{\lambda}_x$  and  $\bar{\lambda}_y$  are proportional to the largest eigenvalues of the matrices  $A$  and  $B$ . For generalized coordinates  $x$  and  $y$  are replaced by  $\xi, \eta$  respectively. This spectral radius is now a function of the preconditioning. Hence,

$$\bar{\lambda}_x = \rho(\mathbf{P}A) \quad \bar{\lambda}_y = \rho(\mathbf{P}B)$$

The coefficients  $\epsilon^{(2)}$  and  $\epsilon^{(4)}$  are adapted to the flow and are defined as follows:

$$\epsilon_{i+\frac{1}{2},j}^{(2)} = \kappa^{(2)} \max(\nu_{i-1,j}, \nu_{i,j}, \nu_{i+1,j}, \nu_{i+2,j}),$$

$$\nu_{i,j} = \frac{|p_{i+1,j} - 2p_{i,j} + p_{i-1,j}|}{|p_{i+1,j} + 2p_{i,j} + p_{i-1,j}|},$$

$$\epsilon_{i+\frac{1}{2},j}^{(4)} = \max \left[ 0, \left( \kappa^{(4)} - \epsilon_{i+\frac{1}{2},j}^{(2)} \right) \right],$$



where  $p$  is the pressure, and the quantities  $\kappa^{(2)}$  and  $\kappa^{(4)}$  are constants to be specified. The operators in (15) for the  $y$  direction are defined in a similar manner.

The second-difference dissipation term is nonlinear. Its purpose is to introduce an entropy-like condition and to suppress oscillations in the neighborhood of shocks. This term is small in the smooth portion of the flow field. The fourth-difference dissipation term is basically linear and is included to damp high-frequency modes and allow the scheme to approach a steady state. Only this term affects the linear stability of the scheme. Near shocks it is reduced to zero. For incompressible flow shocks can only appear in the, nonphysical, transient and so the second-difference dissipation is not important. To reemphasize, the preconditioning matrix multiplies the flux terms but not the artificial viscosity terms. The scaling in the artificial viscosity depends on the spectral radius of the preconditioned matrices. If one were to use a matrix valued viscosity, [14], it would be related to the absolute value of the preconditioned Jacobian matrices.

The boundary conditions at the far field boundary, for subsonic flow, are based on the one dimensional theory of characteristics in the direction normal to the boundary. The preconditioning changes the form of these characteristic variables. They are now given by

$$\begin{aligned} R_1 &= u - \frac{p'}{2\rho\beta^2} \left( u(1 - \alpha - \frac{\beta^2}{c^2}) - \sqrt{(u(1 - \alpha - \frac{\beta^2}{c^2}))^2 + 4(1 - \frac{u^2}{c^2})\beta^2} \right) \\ R_2 &= u - \frac{p'}{2\rho\beta^2} \left( u(1 - \alpha - \frac{\beta^2}{c^2}) + \sqrt{(u(1 - \alpha - \frac{\beta^2}{c^2}))^2 + 4(1 - \frac{u^2}{c^2})\beta^2} \right) \end{aligned}$$

where  $u$  is the component of the velocity normal to the boundary. This formulas simplify slightly if  $\alpha = 1$  and more if  $\alpha = 1 + \frac{\beta^2}{c^2}$ . If we consider low Mach numbers then we can approximate these by

$$R_1 = u - \frac{p'}{\rho\beta}, \quad R_2 = u + \frac{p'}{\rho\beta}$$

which is the same as for the incompressible case. At solid boundaries the normal momentum equation is used which is not affected by the preconditioning.

The solution is advanced by the explicit Runge-Kutta method described above and without any residual smoothing or multigrid. We present two calculations for inviscid flow about a NACA 0012. The first calculation is for inflow conditions  $M = 0.03, \alpha = 4^\circ$ . In this case we see that the residual asymptotes without the use of preconditioning and that the preconditioning dramatically increases the rate of convergence. The use of the preconditioning adds only a few percent to the total computational time. For viscous flows the computational time required for the preconditioning is negligible. In the second case we consider the same geometry but with an inflow of  $M = 0.8, \alpha = 1.25^\circ$ . In this case we also see a increased rate of convergence for the preconditioned case but not as dramatic as before.

In all cases we could not allow  $\beta$  to become too small. In fact the cutoff is sufficiently large so that  $\beta$  is close to a constant. This has been observed in many central difference Runge-Kutta codes but has not been observed in the upwind code coupled with an ADI solver [3].

## 7 Conclusion

We have considered a family of matrix preconditionings for both the incompressible and compressible fluid dynamic equations that generalize previous results. In both cases the wave speeds are

more equalized than for the original set of equations and so the condition number of the system is reduced. For the compressible equations the condition is equal to 1 at a Mach number of zero and increases as the Mach number increases. At  $M = 1$  the condition number is infinite but it increases at a slower rate than for the physical system.

In addition to the question of the convergence rate to a steady state we have considered the question of the accuracy of the numerical scheme for low Mach numbers. One can prove that for the partial differential equation that the compressible equations approach the incompressible equations as the Mach number goes to zero. For the numerical scheme this is no longer generally true and so the accuracy of the numerical scheme to the compressible equations decreases as the Mach number goes to zero. One way to improve the situation is to include the preconditioning in the Riemann solver, or equivalently, to account for the preconditioning in the artificial viscosity. For example, for low Mach numbers the scalar artificial viscosity for the continuity equation should be based on the pressure rather than the density.

## A Appendix

Let  $W$  denote the conservative variables  $(\rho, m, n, E)^t$ , with  $m = \rho u, n = \rho v$ , let  $w$  denote the primitive variables  $(p, u, v, S)^t$  and let  $\hat{w}$  denote  $(p, u, v, T)^t$ . Then

$$\frac{\partial W}{\partial w} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & -\frac{\rho}{c_p} \\ \frac{u}{c^2} & \rho & 0 & -\frac{m}{c_p} \\ \frac{v}{c^2} & 0 & \rho & -\frac{n}{c_p} \\ \frac{1}{\gamma-1} + \frac{M^2}{2} & m & n & -\frac{\rho(u^2+v^2)}{2c_p} \end{pmatrix}$$

$$\frac{\partial w}{\partial W} = \begin{pmatrix} \frac{(\gamma-1)(u^2+v^2)}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{c_p[(\gamma-1)\frac{M^2}{2}-1]}{\rho} & -\frac{(\gamma-1)uc_p}{\rho c^2} & -\frac{(\gamma-1)vc_p}{\rho c^2} & \frac{(\gamma-1)c_p}{\rho c^2} \end{pmatrix}$$

$$\frac{\partial W}{\partial \hat{w}} = \begin{pmatrix} \frac{\rho}{p} & 0 & 0 & -\frac{\rho}{T} \\ \frac{m}{p} & \rho & 0 & -\frac{m}{T} \\ \frac{n}{p} & 0 & \rho & -\frac{n}{T} \\ \frac{E}{p} & m & n & -\frac{\rho(u^2+v^2)}{2T} \end{pmatrix}$$

$$\frac{\partial \hat{w}}{\partial W} = \begin{pmatrix} \frac{(\gamma-1)(u^2+v^2)}{2} & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ -\frac{T}{\rho} + \frac{(\gamma-1)T(u^2+v^2)}{2p} & -\frac{(\gamma-1)u}{\rho R} & -\frac{(\gamma-1)v}{\rho R} & \frac{(\gamma-1)}{\rho R} \end{pmatrix}$$

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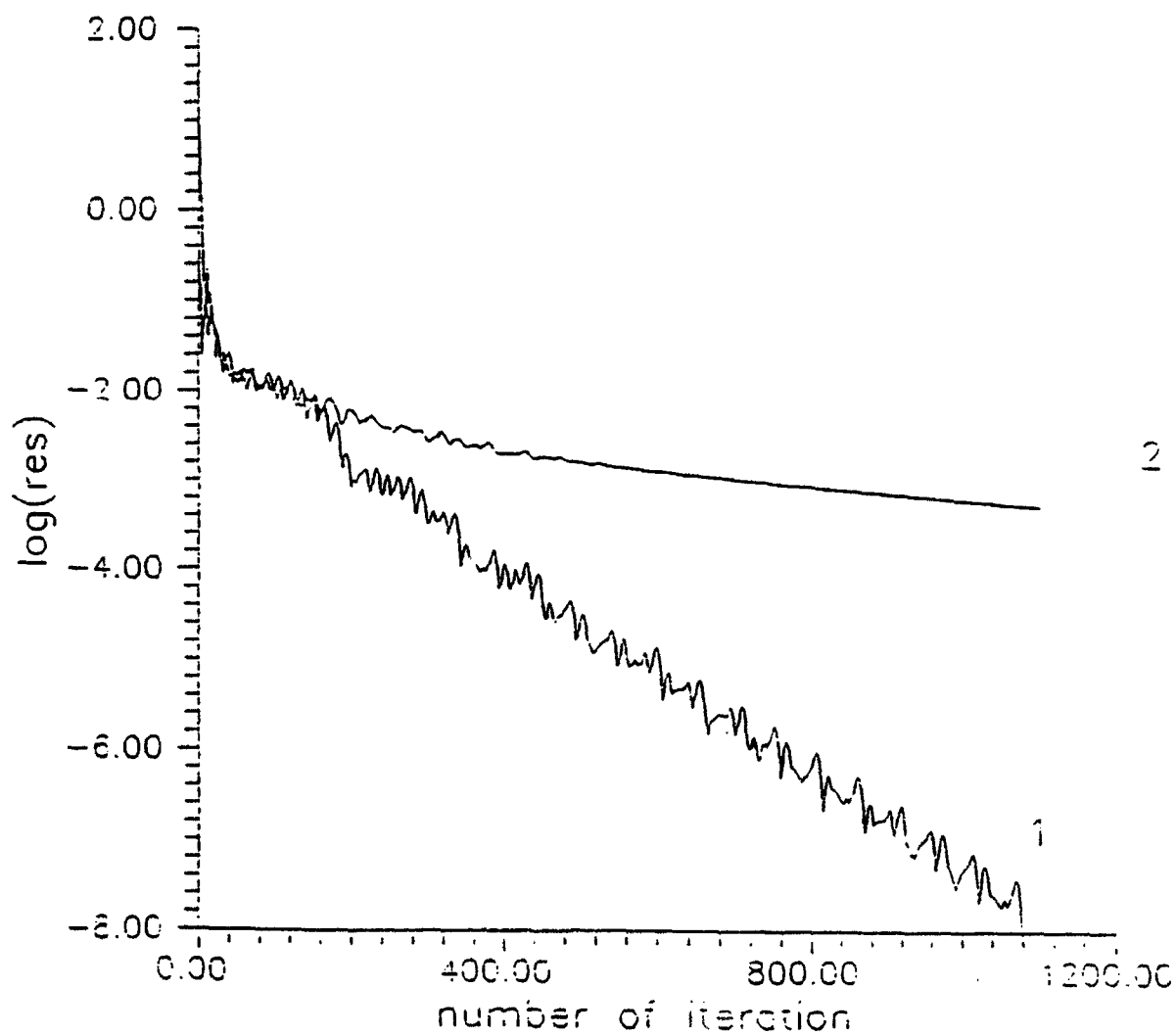


figure 1: Residual for the momentum equation. Flow about a NACA0012,  $M_{\infty} = 0.03$ ,  $\alpha = 4^{\circ}$  with a  $98 \times 22$  C grid. Graph (1) is the preconditioned solution and (2) is without preconditioning.

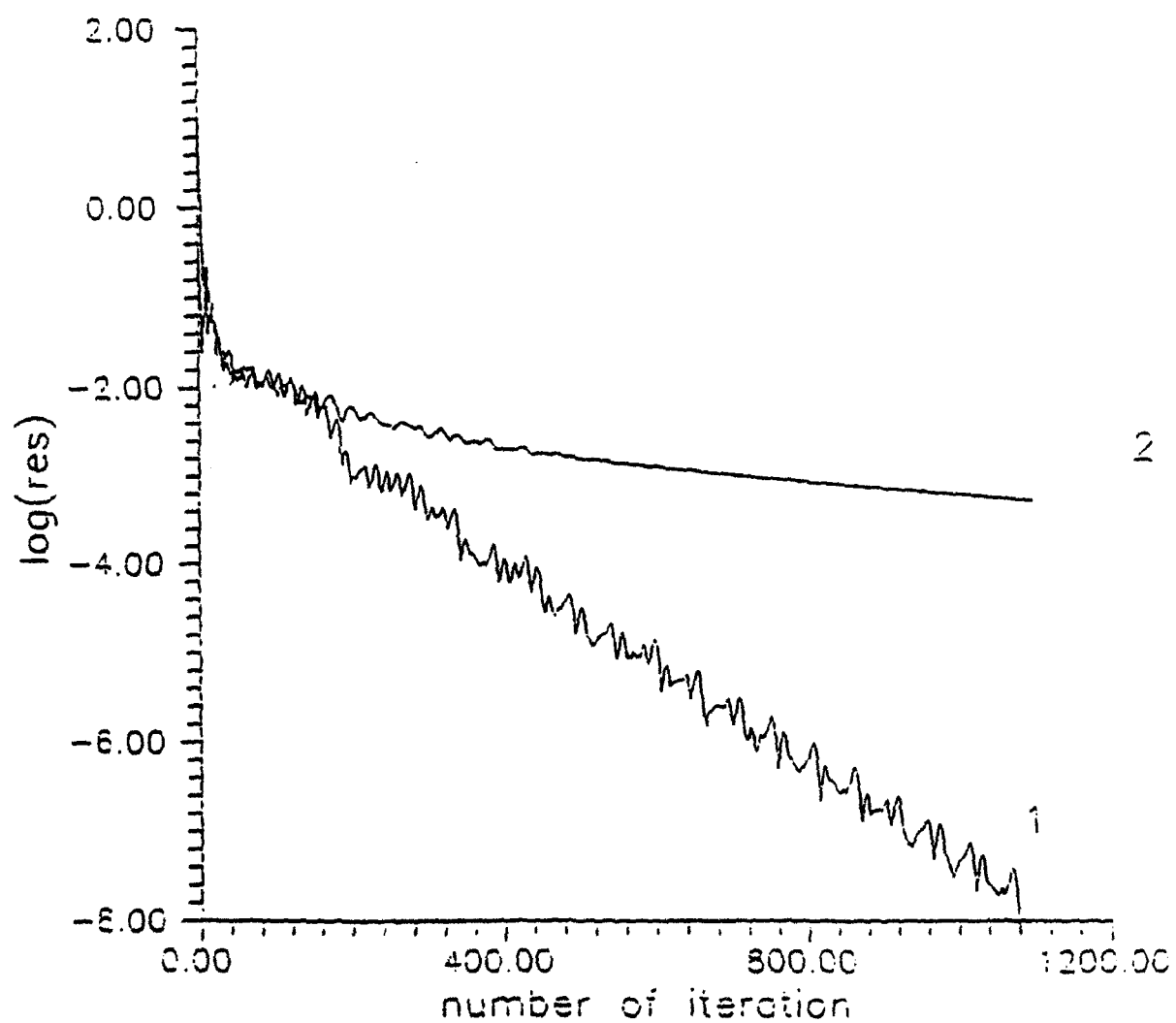


figure 2: Same as figure 1 but with  $M_\infty = 0.80$ ,  $\alpha = 1.25^\circ$ .

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<small>1. REPORT NUMBER (Leave blank) 2. REPORT DATE 3. REPORT TYPE AND DATES COVERED 4. TITLE AND SUBTITLE 5. FUNDING NUMBERS 6. AUTHOR(S) 7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) 8. PERFORMING ORGANIZATION REPORT NUMBER 9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) 10. SPONSORING/MONITORING AGENCY REPORT NUMBER 11. SUPPLEMENTARY NOTES 12a. DISTRIBUTION/AVAILABILITY STATEMENT 12b. DISTRIBUTION CODE 13. ABSTRACT (Maximum 200 words) 14. SUBJECT TERMS 15. NUMBER OF PAGES 16. PRICE CODE 17. SECURITY CLASSIFICATION OF REPORT 18. SECURITY CLASSIFICATION OF THIS PAGE 19. SECURITY CLASSIFICATION OF ABSTRACT 20. LIMITATION OF ABSTRACT</small>				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE July 1993		3. REPORT TYPE AND DATES COVERED Contractor Report
4. TITLE AND SUBTITLE PRECONDITIONING AND THE LIMIT TO THE INCOMPRESSIBLE FLOW EQUATIONS			5. FUNDING NUMBERS C NAS1-18605 C NAS1-19480  WU 505-90-52-01	
6. AUTHOR(S) E. Turkel, A. Fiterman, and B. van Leer				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23681-0001			8. PERFORMING ORGANIZATION REPORT NUMBER  ICASE Report No. 93-42	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Langley Research Center Hampton, VA 23681-0001			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA CR-191500 ICASE Report No. 93-42	
11. SUPPLEMENTARY NOTES  Langley Technical Monitor: Michael F. Card Final Report  Submitted to SIAM Journal on Numerical Analysis				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited  Subject Category 64			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words)  We consider the use of preconditioning methods to accelerate the convergence to a steady state for both the incompressible and compressible fluid dynamic equations. We also consider the relation between them for both the continuous problem and the finite difference approximation. The analysis relies on the inviscid equations. The preconditioning consists of a matrix multiplying the time derivatives. Hence, the steady state of the preconditioned system is the same as the steady state of the original system. For finite difference methods the preconditioning can change and improve the steady state solutions. An application to flow around an airfoil is presented.				
14. SUBJECT TERMS preconditioning, Euler equations			15. NUMBER OF PAGES 22	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT	

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)  
Prescribed by ANSI Std. Z39-18  
298-102